Scattering on Kerr black holes

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Modern Developments in General Relativity and their Historical Roots

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1. Fixed-frequency scalar-wave scattering theory on Kerr

The bulk of the literature on classical black hole scattering-theory for scalar waves on Kerr backgrounds concerns only the *fixed-frequency* study of solutions

$$u_{(\omega,m,\ell)}(r^*)$$

to the radial o.d.e.

$$u'' + \omega^2 u = V u,$$

where V is the potential

$$V = V_{(\omega,m,\ell)}(r^*),$$

resulting from CARTER's remarkable separation of the linear scalar wave equation

$$\Box_g \psi = 0.$$

See CHANDRASEKHAR's monumental monograph.

At the conceptual level, one of the most interesting new phenomena of black hole scattering which arises when passing from the Schwarzschild $a = 0$ to the rotating $a \neq 0$ Kerr case is that of *superradiance*.

For each fixed frequency triple (ω, m, ℓ) with $\omega \in \mathbb{R}$, one can define two complex-valued solutions $U_{\rm hor}(r^*)$ and $U_{\rm inf}(r^*)$ of the radial o.d.e. so that

$$U_{\rm hor} \sim e^{-i(\omega - \omega_+ m)r^*} \text{ as } r^* \to -\infty, \qquad U_{\rm inf} \sim e^{i\omega r^*} \text{ as } r^* \to \infty,$$

corresponding to the asymptotic behaviour of the potential V, which is itself real. Here $2M\omega_+(M + \sqrt{M^2 - a^2}) = a$.

It follows that since $\overline{U_{\rm inf}}$ also solves the radial o.d.e., we may write[a]

$$\overline{U_{\rm inf}} = \frac{\omega}{(\omega - \omega_+ m)} \mathfrak{T} U_{\rm hor} + \mathfrak{R} U_{\rm inf}, \tag{1}$$

where

$$\mathfrak{T} = \mathfrak{T}(\omega, m, \ell), \qquad \mathfrak{R} = \mathfrak{R}(\omega, m, \ell)$$

are known as the *transmission* and *reflexion* coefficients.

[a] If $U_{\rm hor}$ and $U_{\rm inf}$ are linearly independent!

With the above precise normalisation of $\mathfrak{R}$ and $\mathfrak{T}$, the energy identity associated to the radial o.d.e. yields

$$|\mathfrak{R}|^2 + \frac{\omega}{\omega - \upomega_+ m}|\mathfrak{T}|^2 = 1.$$

Superradiance, first discussed by ZELDOVICH, corresponds to the fact that, for the frequency range

$$\omega(\omega - \upomega_+ m)^{-1} < 0, \tag{2}$$

we have

$$|\mathfrak{R}(\omega, m, \ell)| > 1. \tag{3}$$

That is to say, there is a nontrivial energy amplification factor at fixed frequency.

The above discussion required that U_{hor} and U_{inf} are linearly independent, i.e., that U_{hor} does not represent a "resonance" with real frequency $\omega \in \mathbb{R}$.

Were $U_{\mathrm{hor}}(\omega_0, m, \ell)$ to represent a resonance, then $\mathfrak{R}(\omega, m, \ell)$ would blow up as $\omega \to \omega_0$ is approached.

Surprisingly, this very basic question was only resolved recently:

Theorem (Y. SHLAPENTOKH-ROTHMAN, 2013)**.** *There are no resonances on the real axis. For all* $\omega \in \mathbb{R}$ *and* m, ℓ*, the solutions* U_{hor} *and* U_{inf} *are indeed linearly independent and thus* $\mathfrak{R}(\omega, m, \ell)$ *is well-defined and finite.*

cf. WHITING's celebrated mode stability which showed that there are no finite-energy modes for $Im(\omega) > 0$.

What is it about the Kerr spacetime which makes mode stability true?

This remains mysterious!

Theorem (G. MOSCHIDIS, 2016)**.** *One can construct axisymmetric stationary spacetimes* $(\mathcal{M}, g)$ *coinciding with Kerr in the ergoregion and near infinity but for which there* **does exist** *a resonance on the real axis or in the upper half plane.*

These can be thought of as "designer superradiant instabilities".

Turning back to Kerr, SHLAPENTOKH-ROTHMAN's "mode stability on the real line" theorem shows that there indeed exists a well-defined fixed-frequency scattering theory with finite reflection coefficients $\mathfrak{R}$.

What the above does not resolve is whether the reflection coefficients are uniformly bounded in frequency, i.e. whether the constant

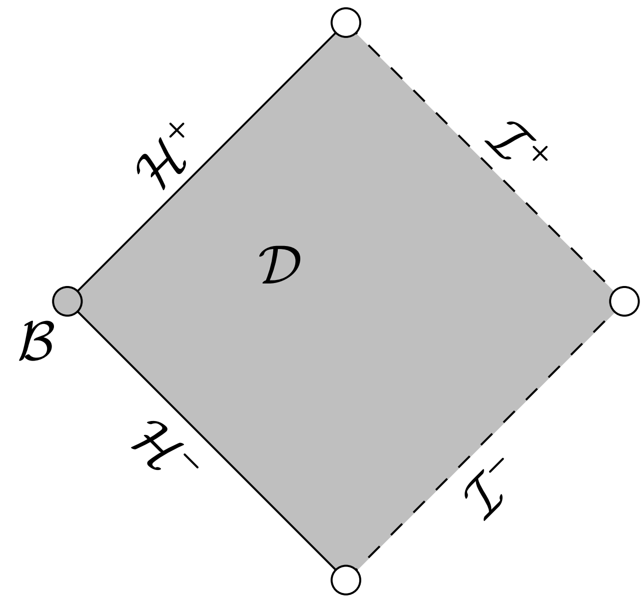
$$S(a, M) \doteq \sup_{(\omega,m,\ell)} |\mathfrak{R}(\omega, m, \ell)|$$

is finite (cf. STAROBINSKII).

This potential "ultraviolet instability" is a fundamental obstacle in passing from the formal fixed-frequency scattering theory to a genuine time-domain scattering theory of finite-energy wave packets.

2. The set-up for scalar-wave scattering theory in the time-domain

We will fix subextremal Kerr parameters $|a| < M$ and consider the Kerr metric $g_{a,M}$ defined on a "domain of outer communications" $\mathcal{D}$.



The boundary components $\mathcal{H}^\pm$ correspond to past and future event horizons and meet in the so-called bifurcation sphere $\mathcal{B}$. We can also associate two "asymptotic" boundary components future and past null infinity $\mathcal{I}^\pm$.

Cauchy problem for smooth data

Define

$$\Sigma = \{t = 0\} \cup \mathcal{B},$$

where t is the usual Boyer-Lindquist coordinate.

We can consider smooth compactly supported data $(\psi_\Sigma, \psi'_\Sigma)$ on Σ for the wave equation

$$\Box_g \psi = 0.$$

We shall call the map from smooth initial data to solution *forward evolution*:

$$(\psi, \psi') \mapsto \psi. \tag{4}$$

We have

Proposition. *If data* (ψ, ψ') *are smooth of compact support on* Σ*, then the solution* $r\psi$ *extends to a smooth function* ϕ *on* $\mathcal{I}^+$*.*

The "restriction" $\upphi$ of $r\phi$ to $\mathcal{I}^+$ given by the above Proposition is known as the *Friedlander radiation field*.

Trivially, we may also define the radiation field on the horizon $\mathcal{H}^+$ to be the restriction $\uppsi \doteq \psi|_{\mathcal{H}^+}$.

To summarise, forward evolution gives rise to a map on smooth compactly supported initial data

$$(\uppsi|_\Sigma, \uppsi'|_\Sigma) \mapsto \psi \mapsto (\uppsi \doteq \psi|_{\mathcal{H}^+}, \upphi \doteq r\psi|_{\mathcal{I}^+}). \tag{5}$$

The forward maps of our scattering theory will be constructed by completing the above map with respect to suitably defined energies. The states defining scattering theory are associated to energies, defined in turn by vector fields.

Vector field energies and their associate Hilbert spaces

Recall that a general vector field X defines an energy current $\mathbf{J}^X[\psi]$ and an energy flux

$$\int_{\mathcal{S}} \mathbf{J}^X[\psi] \tag{6}$$

through an arbitrary hypersurface $\mathcal{S}$.

For appropriate causal vector fields X for which (6) is *nonnegative*, the inner product induced by the expression (6) defines a Hilbert space

$$\mathcal{E}^X_\Sigma$$

by completion of the set of smooth compactly supported data (ψ, ψ') on Σ.

Similarly, the flux (6) for $\mathcal{S} = \mathcal{H}^+, \mathcal{I}^+$ defines Hilbert spaces of asymptotic states

$$\mathcal{E}^X_{\mathcal{H}^+} \oplus \mathcal{E}^X_{\mathcal{I}^+},$$

via completion of the set of radiation fields arising from the forward map.

In this picture, the problem of scattering theory translates into finding an appropriate causal vector field X so that the forward evolution map on smooth data

$$(\psi|_\Sigma, \psi'|_\Sigma) \mapsto (\psi|_{\mathcal{H}^+}, \phi|_{\mathcal{I}^+}).$$

induces a bounded map of Hilbert spaces

$$\mathcal{E}^X_\Sigma \rightleftarrows \mathcal{E}^X_{\mathcal{H}^+} \oplus \mathcal{E}^X_{\mathcal{I}^+}.$$

which is invertible*.*

3. The T-energy theory and its limitations

The Schwarzschild $a = 0$ case

In the Schwarzschild case $a = 0$, the stationary Killing field T (corresponding to ∂_t in standard coordinates) is timelike in the interior of $\mathcal{D}$ becoming null on $\mathcal{H}^+ \cup \mathcal{H}^-$ and vanishing on $\mathcal{B}$. Thus the *energy density* defined by T degenerates pointwise.

Nonetheless, the completions $\mathcal{E}^T_\Sigma$, $\mathcal{E}^T_{\mathcal{H}^+}$ and $\mathcal{E}^T_{\mathcal{I}^+}$ define Hilbert spaces.

One can indeed show relatively easily that the forward map on smooth compactly supproted Cauchy data extends to a **unitary** isomorphism

$$\mathcal{E}^T_\Sigma \cong \mathcal{E}^T_{\mathcal{H}^+} \oplus \mathcal{E}^T_{\mathcal{I}^+}. \tag{7}$$

DIMOCK–KAY (see also NICOLAS).

The case $a \neq 0$ and the ergoregion

Turning to the Kerr case $a \neq 0$, there is now a non-empty subset $\mathcal{S}$ of $\mathcal{D}$, known as the *ergoregion*, where T is spacelike. In particular, the energy-fluxes

$$\int_{\Sigma} \mathbf{J}^T[\psi], \qquad \int_{\mathcal{H}^+} \mathbf{J}^T[\psi]$$

defined by T fail to be nonnegative. This is the true physical-space origin of the phenomenon of superradiance. Thus we can have

$$\int_{\mathcal{I}^+} \mathbf{J}^T[\psi] > \int_{\Sigma} \mathbf{J}^T[\psi] \qquad (8)$$

and, in fact, *a priori* the left hand side of (8) can be **infinite**.

Part of the conceptual difficulty of formulating a scattering theory for Kerr is thus to find the correct notion of states which replaces those based on $\mathcal{E}^T$.

At the same time, one must understand what property replaces the notion of unitarity as a means of quantifying the good properties of the scattering map.

4. A scattering theory for Kerr: the main theorems

The N-energy theory

The first candidate replacement for the (degenerate) Schwarzschild T-energy is the so-called N-energy.

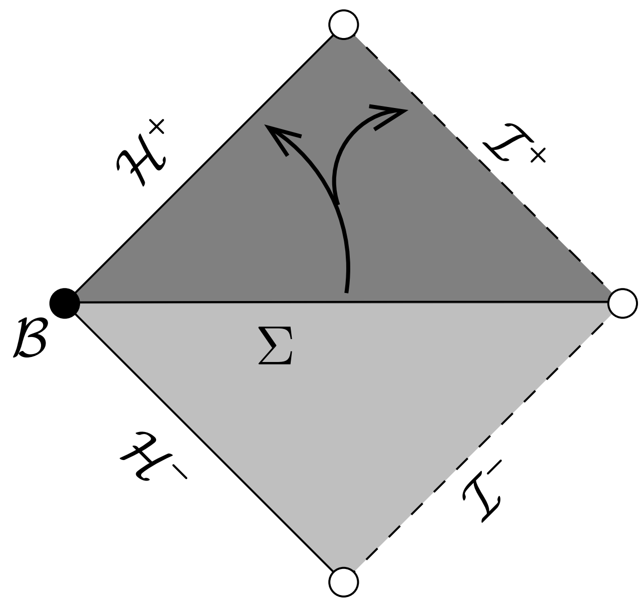
Here, N is a globally timelike vector field which is T-invariant outside a neighbourhood of the bifurcation sphere $\mathcal{B}$ and moreover such that $N = T$ in a neighbourhood of $\mathcal{I}^+$.

The energies $\int \mathbf{J}^N$ associated to this vector field are indeed manifestly positive-definite and pointwise non-degenerate. The energy is no longer conserved however.

The N-energy forward map

The first main theorem defines asymptotic states for all solutions arising from finite N-energy data on the hypersurface Σ for all solutions parametrised by $\mathcal{E}^N_\Sigma$.

Theorem 1. *Forward evolution with data on Σ extends to a bounded map* $\mathscr{F}_+ : \mathcal{E}^N_\Sigma \to \mathcal{E}^N_{\mathcal{H}^+} \oplus \mathcal{E}^T_{\mathcal{I}^+}$.



The above theorem is essentially a corollary of a sequence of previous works of ours (M.D.–RODNIANSKI AND M.D.–RODNIANSKI–SHLAPENTOKH-ROTHMAN) on the Cauchy problem.

The fundamental point is that the non-degenerate N-energy of a solution of the wave equation at all times is bounded by the initial N-energy times a constant *depending only on the Kerr parameters.*

- In the high frequency regime, to obtain this bound one must understand the interaction of the difficulties of superradiance and "trapped null geodesics". Quite fortuitously, *"trapped" frequencies are never superradiant.*

- The **red-shift** effect, which is seen by the vector field N, plays a helpful stabilising role.

- Moreover, direct appeal to (a quantitative refinement of) SHLAPENTOKH-ROTHMAN'S mode stability is necessary to deal with low frequencies.

Interestingly, all these helpful properties degenerate as $|a| \to M$ (cf. ARETAKIS).

A blue-shift instability and the non-existence of an N-energy backwards map

Satisfactory though the forward theory may be, it turns out that finiteness of the above N-energy is ill-suited for defining the states for a scattering theory.

The fundamental origin of this is that the red-shift effect on the horizon–*so favourable for controlling forward evolution!*–under backwards evolution is now seen as a **blue-shift**. This fact is familiar from the calculation typically done to derive Hawking radiation.

One can in fact prove that the map of Theorem 1 fails to be surjective:

Theorem 2. *The map $\mathscr{F}_+$ of Theorem 1 fails to be surjective.*

It follows that there does not exist even a *one-sided* inverse $\mathscr{B}_-$ satisfying

$$\mathscr{F}_+ \circ \mathscr{B}_- = Id.$$

Thus, *existence of scattering states* does not hold in the N-theory.

Aside

Let us note that by introducing sufficiently high exponential weights in the spaces defining the scattering data, i.e. considering the spaces $\mathcal{E}^{e^{\alpha v_+}N}_{\mathcal{H}^+}$ and $\mathcal{E}^{e^{\alpha u_+}T}_{\mathcal{I}^+}$, then there indeed exists a bounded one-sided inverse

$$\mathscr{B}_- : \mathcal{E}^{e^{\alpha v_+}N}_{\mathcal{H}^+} \oplus \mathcal{E}^{e^{\alpha u_+}T}_{\mathcal{I}^+} \to \mathcal{E}^N_\Sigma \tag{9}$$

such that $\mathscr{F}_+ \circ \mathscr{B}_- = id$. Thus, we do have existence of a *restricted class* of future scattering states.

With this setting, we can prove a stronger version of the above theorem saying that $e^{\alpha v_+}$ above cannot be replaced by v_+^p no matter how large p is taken, i.e. the range of $\mathscr{F}_+$ does not contain

$$\mathcal{E}^{v_+^p N}_{\mathcal{H}^+} \oplus \{0\} \qquad \text{or} \qquad \{0\} \oplus \mathcal{E}^{u_+^p N}_{\mathcal{I}^+}.$$

The question of the precise characterization of the range of $\mathscr{F}_+$ remains open.

The V-energy theory

To define a forward map which one can indeed hope to show is invertible, we must pass to a degenerate energy class which does not see the red-shift at the horizon.

Recall that $g_{M,a}$ admits an additional Killing vector field Φ corresponding to axisymmetry. Although for $a \neq 0$, the vector field T fails to be globally timelike in the interior of $\mathcal{D}$, the span of T and Φ does form a timelike plane, and the Killing combination $K = T + \omega_+ \Phi$ is timelike in a neighbourhood of $\mathcal{H}^+$, becoming null on $\mathcal{H}^+$ itself. *(Note that if $a = 0$, then $K = T$, but if $a \neq 0$, then K itself is spacelike away from the axis of symmetry near $\mathcal{I}^+$.)*

We define a T-invariant vector field V with the property that

$$V = K \quad \text{near } \mathcal{H}^+, \qquad V = T \quad \text{near } \mathcal{I}^+$$

and V is timelike in the interior of $\mathcal{D}$. The energy associated to this vector field is manifestly non-negative definite, though degenerate analogous to the T-energy in the Schwarzschild case. (In the case $a \neq 0$, there is necessarily a region where V fails to be Killing.)

Our third main theorem is a degenerate V-energy analogue of Theorem 1:

Theorem 3. *Forward evolution extends to a bounded map*

$$\mathscr{F}_+ : \mathcal{E}^V_\Sigma \to \mathcal{E}^K_{\mathcal{H}^+} \oplus \mathcal{E}^T_{\mathcal{I}^+}.$$

To show this statement one must show that the degenerate V-energy is bounded by its initial value times a constant *depending only on the Kerr parameters*.

This requires us to revisit our pevious boundedness theorem for the Cauchy problem on Kerr, reworking the arguments so as to depend only on the degenerate energy, *i.e. not exploiting the helpful red-shift effect*.

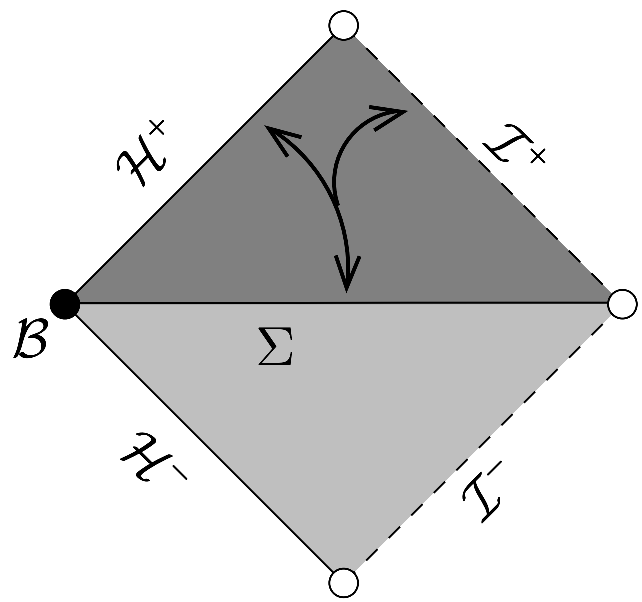
The V-energy backwards map

Our degenerate-energy class is indeed suitable to construct a bounded inverse of the map of Theorem 3:

Theorem 4. *There exists a bounded map*

$$\mathscr{B}_- : \mathcal{E}^K_{\mathcal{H}^+} \oplus \mathcal{E}^T_{\mathcal{I}^+} \to \mathcal{E}^V_{\Sigma}\,, \tag{10}$$

inverting the map of Theorem 3, i.e. $\mathscr{B}_- \circ \mathscr{F}_+ = Id$ *and* $\mathscr{F}_+ \circ \mathscr{B}_- = Id$.



Existence and boundedness of the S-matrix

First, note that applying a discrete isometry of $\mathcal{D}$ which interchanges the future and past of Σ, we infer analogously to Theorems 3 and 4 the existence of a bounded <u>past</u> forward map and bounded two-sided inverse

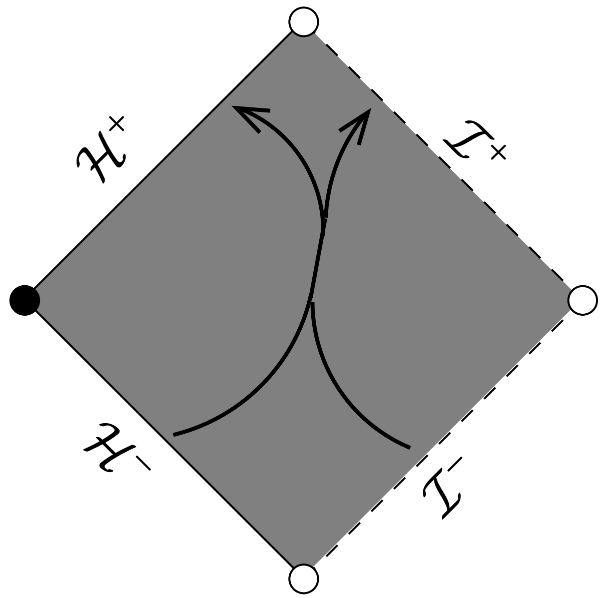
$$\mathscr{F}_- : \mathcal{E}^V_\Sigma \to \mathcal{E}^K_{\mathcal{H}^-} \oplus \mathcal{E}^T_{\mathcal{I}^-}, \qquad \mathscr{B}_+ : \mathcal{E}^K_{\mathcal{H}^-} \oplus \mathcal{E}^T_{\mathcal{I}^-} \to \mathcal{E}^V_\Sigma .$$

We thus have both existence and uniqueness for *past* scattering states as well as *past* asymptotic completeness.

The following is then an immediate corollary

Corollary. *The composition of* $\mathscr{S} = \mathscr{F}_+ \circ \mathscr{B}_+$ *defines a bounded invertible map*

$$\mathscr{S} : \mathcal{E}^K_{\mathcal{H}^-} \oplus \mathcal{E}^T_{\mathcal{I}^-} \to \mathcal{E}^K_{\mathcal{H}^+} \oplus \mathcal{E}^T_{\mathcal{I}^+}. \tag{11}$$



The boundedness $\|\mathscr{S}\| \leq C$ of the map $\mathscr{S}$ in the operator norm is the correct quantitative replacement for the usual unitarity property.

A physical space theory of superradiant reflection

Given the scattering map $\mathscr{S}$, we can now give an account of superradiant reflection in physical space, i.e. in the "time domain".

Recall the standard physical set-up: One wishes to study the scattering of waves with no ingoing contribution from the past event horizon $\mathcal{H}^-$. We thus pass from $\mathscr{S}$ to the transmission map $\mathscr{T}$ and reflection map $\mathscr{R}$ defined by

$$\mathscr{S}\big|_{\{0\}\oplus\mathcal{E}^T_{\mathcal{I}^-}} = \mathscr{T} \oplus \mathscr{R}$$

where

$$\mathscr{R} : \mathcal{E}^T_{\mathcal{I}^-} \to \mathcal{E}^T_{\mathcal{I}^+}, \qquad \mathscr{T} : \mathcal{E}^T_{\mathcal{I}^-} \to \mathcal{E}^K_{\mathcal{H}^+}.$$

The boundedness of $\mathscr{S}$ above immediately yields the strictly weaker statement

Theorem 5. *The reflection and transmission maps $\mathscr{R}$ and $\mathscr{T}$ are bounded, i.e. $\|\mathscr{R}\|, \|\mathscr{T}\| \leq C$.*

We can show that our physical space operators $\mathscr{R}$ and $\mathscr{T}$ are related to our fixed frequency reflection coefficients $\mathfrak{R}$ and $\mathfrak{T}$ by a generalised Fourier transform:

Theorem 6. *We may represent*

$$\mathscr{R}[\phi] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\ell} a_{\mathcal{I}^-} \mathfrak{R}\, e^{-iu\omega} e^{im\phi} S_{m\ell}(a\omega, \cos\theta)\, d\omega$$

and

$$\mathscr{T}[\phi] = -\frac{1}{\sqrt{4M\pi r_+}} \int_{-\infty}^{\infty} \sum_{m\ell} \left(\frac{\omega}{\omega - \omega_+ m} \right) a_{\mathcal{I}^-} \mathfrak{T}\, e^{-iv\omega} e^{im\phi} S_{m\ell}(a\omega, \cos\theta)\, d\omega.$$

Here

$$-i\omega a_{\mathcal{I}^-} \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \partial_t \phi\, e^{iv\omega} e^{-im\phi} S_{m\ell}(a\omega, \cos\theta)\, \sin\theta\, dv\, d\theta\, d\phi.$$

In particular, we have

$$\sup_{(\omega, m, \ell)} |\mathfrak{R}(\omega, m, \ell)| = \|\mathscr{R}\|.$$

We immediately infer

Corollary. *The reflection and transmission coefficients are indeed uniformly bounded over all frequencies:*

$$\sup_{(\omega,m,\ell)} |\mathfrak{R}(\omega,m,\ell)| \le C, \qquad \sup_{(\omega,m,\ell)} |\mathfrak{T}(\omega,m,\ell)| \le C. \tag{12}$$

This resolves thus the question we started with.

Since $\mathfrak{R}(\omega,m,\ell) > 1$ for any superradiant frequency we have:

Corollary. *For $a \neq 0$, the reflection map $\mathscr{R}$ has norm strictly greater than 1, i.e. we have $C > \|\mathscr{R}\| > 1$.*

The above corollary can be viewed as the definitive time-domain interpretation of the phenomenon of superradiant reflection.

Uniqueness of scattering states for ill-posed scattering data

Finally, we note that our scattering theory allows us to make the following injectivity statements which can be understood as statements just of *uniqueness* of scattering states for scattering data determined on any of the four "ill-posed" pairs of asymptotic boundaries $\mathcal{H}^+ \cup \mathcal{H}^-$, $\mathcal{I}^+ \cup \mathcal{I}^-$, $\mathcal{H}^+ \cup \mathcal{I}^-$ and $\mathcal{H}^- \cup \mathcal{I}^+$.

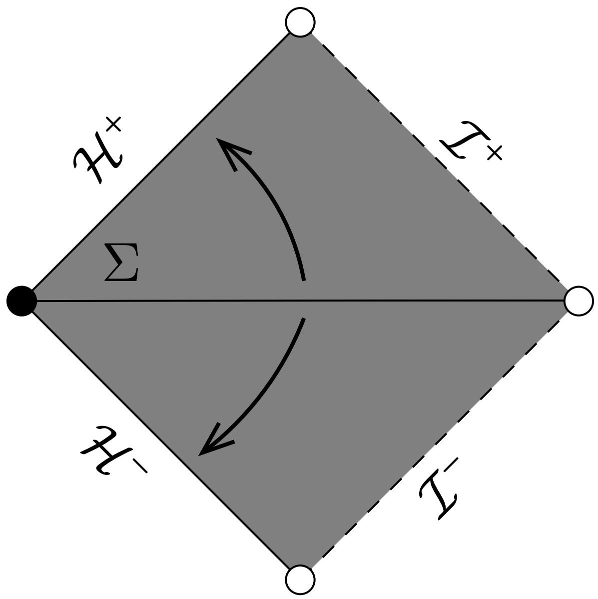
Theorem 7. *The maps*

$$\mathscr{F} : \mathcal{E}^V_\Sigma \to \mathcal{E}^K_{\mathcal{H}^+} \oplus \mathcal{E}^K_{\mathcal{H}^-}, \qquad \mathscr{F} : \mathcal{E}^V_\Sigma \to \mathcal{E}^T_{\mathcal{I}^+} \oplus \mathcal{E}^T_{\mathcal{I}^-}$$

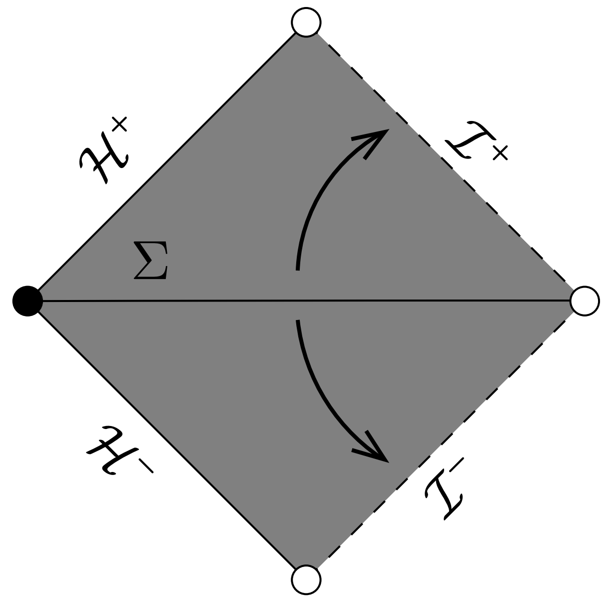
$$\mathscr{F} : \mathcal{E}^V_\Sigma \to \mathcal{E}^K_{\mathcal{H}^+} \oplus \mathcal{E}^T_{\mathcal{I}^-}, \qquad \mathscr{F} : \mathcal{E}^V_\Sigma \to \mathcal{E}^K_{\mathcal{H}^-} \oplus \mathcal{E}^T_{\mathcal{I}^+}$$

are all injective.

$$\mathscr{F} : \mathcal{E}^{V}_{\Sigma} \to \mathcal{E}^{K}_{\mathcal{H}^+} \oplus \mathcal{E}^{K}_{\mathcal{H}^-}$$



$$\mathscr{F} : \mathcal{E}^{V}_{\Sigma} \to \mathcal{E}^{T}_{\mathcal{I}^+} \oplus \mathcal{E}^{T}_{\mathcal{I}^-}$$



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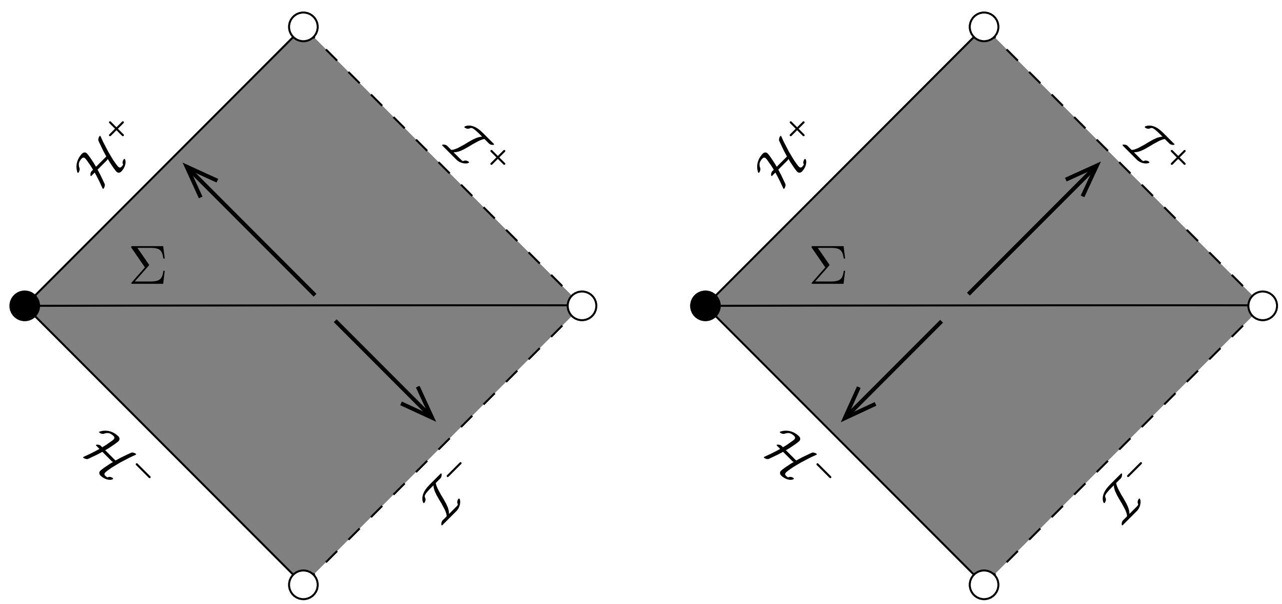

Together with the previous results, the above implies that finite V-energy solutions are uniquely determined by their fluxes to any pair of the set $\{\mathcal{H}^+, \mathcal{H}^-, \mathcal{I}^+, \mathcal{I}^-\}$.

In contrast, however, to the forward maps of Theorem 3, it follows already from general local ill-posedness type results for the wave equation (see e.g. the classic textbook of HADAMARD) that the above maps $\mathscr{F}$ are not surjective. Thus, one does not have the analogue of "existence of scattering states" (cf. (a)) for scattering states parameterized as above.

This is of course in sharp distinction to the fixed-frequency theory, for which "existence of scattering states" associated to $\mathcal{H}^+ \cup \mathcal{H}^-$ *and* $\mathcal{I}^+ \cup \mathcal{I}^-$*, respectively, corresponds precisely to the existence and linear independence of the pairs* $U_{\mathrm{hor}}, \overline{U}_{\mathrm{hor}}$ *or alternatively* $U_{\mathrm{inf}}, \overline{U}_{\mathrm{inf}})$ *described in the beginning of this introduction, on which the whole theory is based.*

5. Epilogue: taming blue-shift instabilities and the relation with black hole interiors

We make a few comments on scattering theory for non-linear generalisations of the scalar wave equation $\Box_g \psi = 0$.

Perhaps the ultimate nonlinear such generalisation is provided by the Einstein vacuum equations

$$\mathrm{Ric}(g) = 0 \tag{13}$$

themselves, where the background geometry is now itself unknown.

The problem of characterizing *all* “admissible” solutions by appropriate asymptotic scattering states may turn out to be too ambitious for equations as nonlinear as (13).

The construction of a restricted class, however, can serve as an important way of obtaining non-trivial examples of solution spacetimes which cannot otherwise easily be inferred to exist.

A result in that direction has recently been provided by

Theorem (M.D.–G. HOLZEGEL–I. RODNIANSKI). *Consider asymptotic data on* $\mathcal{H}^+ \cup \mathcal{I}^+$ *for the Einstein vacuum equations, decaying towards exact Kerr data corresponding to* $g_{a,M}$ *with* $|a| \leq M$ *at a sufficiently fast* <u>*exponential*</u> *rate. Then there exists a vacuum* $(\mathcal{M}, g)$*, with regular horizon* $\mathcal{H}^+$*, attaining the data.*

The spacetimes $(\mathcal{M}, g)$ constructed in the above are in fact the first known examples of dynamical vacuum black holes settling down to Kerr.

The above theorem can be thought of as a non-linear analogue of the map

$$\mathscr{B}_- : \mathcal{E}^{e^{\alpha v_+} N}_{\mathcal{H}^+} \oplus \mathcal{E}^{e^{\alpha u_+} T}_{\mathcal{I}^+} \to \mathcal{E}^N_\Sigma$$

We note that the special structure present in the nonlinearities of the Einstein vacuum equations is <u>essential</u> for the proof. The analogue of the above theorem does <u>not</u> hold even say for the general scalar semilinear equation of the form

$$\Box_g \psi = Q(\nabla\psi, \nabla\psi).$$

Enforcing exponential decay to Kerr is perhaps undesirable because generic solutions of the Cauchy problem are expected to decay only inverse polynomially.

But we already know (from our previous aside) that at least for the scalar wave equation, were we to *start* with *generic scattering data* in

$$\mathcal{E}^{v_+^p N}_{\mathcal{H}^+} \oplus \mathcal{E}^{u_+^p T}_{\mathcal{I}^+}$$

and solve *backwards*, we would generically arrive at something which is not in $\mathcal{E}^N_\Sigma$, i.e. which is singular at the horizon.

For the "generic" non-linear equation, because of backreaction, this would mean that one would expect **not to be able to solve at all** the backwards problem for such data.

Stability of the Kerr Cauchy horizon

For the Einstein vacuum equations things are much better. Indeed, the situation is very similar with what happens for forward evolution in the interior of black holes.

Theorem (M.D.–J. LUK). *Consider arbitrary characteristic initial data on a bifurcate horizon* $\mathcal{H}_A^+ \cup \mathcal{H}_B^+$ *such that the data are globally close to and in fact approach Kerr with* $0 < |a| < M$ *at an inverse polynomial rate. Then, just as in exact Kerr,* **the vacuum spacetime $(\mathcal{M}, g)$ evolving from data is bounded to the future by a bifurcate Cauchy horizon accross which g is continuously extendible.** *The proof leaves open the possibility that Christoffel symbols fail to be square integrable at the Cauchy horizon, making it an essential null singularity.*

Corollary. *If Kerr is indeed non-linearly stable in the exterior, then its causal structure is stable in the interior. In particular, spacetimes arising from data sufficiently close to Kerr data will not form spacelike singularities.*

Event horizon-singular spacetimes

The above theorem on the nonlinear stability of a (possibly singular) Cauchy horizon (together with the existence of a degenerate V-scattering theory for the linear wave equation) gives support to the following conjecture:

Conjecture. *Consider asymptotic data on $\mathcal{H}^+ \cup \mathcal{I}^+$ as before but which decay to $g_{a,M}$ only at a sufficiently fast inverse* polynomial *rate. Then there exists a vacuum spacetime $(\mathcal{M}, g)$ attaining the data. For generic such data, $\mathcal{H}^+$ is a "weak null singularity" across which the metric extends continuously but with Christoffel symbols which fail to be locally square integrable.*

Lesson: When solving the Einstein equations **forward** from smooth initial data, asymptotic data on $\mathcal{H}^+$ and $\mathcal{I}^+$ decay polynomially *but are also correlated*. If one starts with generic uncorrelated "scattering" data on $\mathcal{H}^+$ and $\mathcal{I}^+$, one can indeed solve **backwards** and construct a dynamic black hole spacetime, but the event horizon will turn into an essential null singularity.